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J.-H. Lambert's theory of probable syllogisms

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ABSTRACT

In his *Neues Organon* of 1764, the mathematician and astronomer Jean-Henri Lambert [12] developed a theory of probable syllogisms, with the aim of formally describing the probabilist reasoning and then applying it to the probability of testimony, thus imparting an epistemic sense to the concept of probability. In the present text, we propose to study the principles of this theory and the difficulties it raises.

1. Introduction

Our familiarity with probabilistic reasoning might lead us to believe that its acquisition presents no major obstacles, although it does of course require a certain amount of training. In reality, the transition from a "logic of the true" to a "logic of the probable" is not simply a matter of taking into account an extra degree of complexity. It implies a conversion of our relation to the object of knowledge, which does not receive *one value* of truth, but is attributed a *degree* of belief. The difficulty we may encounter in reasoning in this manner becomes particularly apparent when the probabilist reasoning is conducted in a framework initially suited to a logic of the true, namely syllogistic logic. And yet this is precisely the situation we find in the history of the creation of probability calculus during the first half of the 18th century, particularly in the work of Johann-Heinrich Lambert (1728–1777), who set out the principles of a theory of probable syllogisms. He went even further, by drawing up the rules for calculating the degree of certainty of a conclusion on the basis of probable hypotheses, or as he put it, the "first elements of this calculus" (*die ersten Gründe dieser Berechnung*).

Our intention here is to study the way Lambert constructed his probabilistic syllogistics, with a view to determining which representation of the probabilistic approach is involved and to what degree the syllogistic form is appropriate to it.

2. The context of J.-H. Lambert's probabilistic syllogistics

Jean-Henri Lambert (1728–1777) was a particularly fertile mathematician and astronomer from Mulhouse, whose works on photometry and perspective in particular were of decisive importance. His mathematical works go well beyond the proof of the irrationality of π. He also had an important influence on the construction of Kant's philosophy. His contributions to probability theory[1] were not limited to probabilistic logic: they also cover the theory of errors (the maximum likelihood method) and demographic studies [13].[2]

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[1] The probabilistic dimension of Lambert's work is presented in Sheynin [21] and Shafer [22,23].
[2] On Lambert's analysis of mortality, see the article by Barbut et al. [2].

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The project of constructing a logic of the probable incorporating developments in mathematical probability theory predates the work of Lambert. It was inchoate in the last chapter of *Logique ou l'art de penser* by Antoine Arnauld and Pierre Nicole ([1]), and was treated by Jacques Bernoulli in *Ars conjectandi* (1713). The originality of Lambert's work lies in his construction of a probabilistic syllogistics in which he set out the principles of a theory of probable syllogisms. He was not the first to venture down this path, as the algebraist Gabriel Cramer had already attempted something similar, although with rather less success.

Nowadays, Gabriel Cramer (1704–1752), a Swiss mathematician who was professor at the Academy of Geneva, is best known for his work on algebraic curves and systems of linear equations. During the years 1744–1745 he gave a course of private lessons on logic, of which a large part (paragraphs 448–547, covering 88 pages) was devoted to probabilities.[3] Although Cramer's analysis does use the syllogistic form, it is limited to attributing an arbitrary degree of certainty to each of the two premises, in order to then calculate the degree of certainty of the conclusion by applying the rule of compound probabilities.[4]

Lambert [12] propounds his theory of probable syllogisms in part 4 of *Neues Organon*, published in 1764. The title of this work is clearly a double historical reference:

- Reference to Aristotle, whose six treaties presenting the principles and rules of logic, and notably the theory of the syllogism, were grouped together under the collective title *Organon*.
- Reference to Francis Bacon, whose *Novum Organum* (1620) described a system of inductive logic.

So the project Lambert had at heart in this work was to renew Aristotelian logic. However, this was not simply a matter of completing Aristotle's logic by the addition of a system of inductive logic, but of reforming it, and in two ways. Firstly, Lambert assumed the Leibnizian heritage of *mathesis universalis*, and the requirement of mathematical rigour that it entails. Secondly, he incorporated into the logic a theory of appearance (*Schein*), the function of which is to provide us with the means to go beyond the appearance and reach the true. Appearance here denotes both the modality by which the object of knowledge presents itself to the view of the knowing subject, and an intermediary between the true and the false. This doctrine of appearance, or phenomenology, provides the framework for the probabilistic logic presented in Section 5, entitled "Von dem Warscheinlichkeit" ("On the probable"). Even so, the break with Aristotelian tradition is not total, since the probabilistic logic developed by Lambert takes a syllogistic form.

3. Forms of probability

Lambert starts by distinguishing between two types of proofs according to the method of demonstration used and to the nature of the certainty to which they lead. The purpose of this distinction is to separate the field of the probable from the field of the true. On the one hand, geometric certainty (*geometrische Gewißheit*) is obtained by deduction, following a rigorous sequence of propositions. Moral certainty (*moralische Gewißheit*), on the other hand, is produced by means of a series of arguments. It is the use of *arguments* that characterises moral proofs (*moralische Beweise*) in relation to demonstrations (*Demonstrationen*). Consequently, the concept of moral certainty receives a double meaning: either it denotes a lower level of certainty compared to geometric certainty, because it is not generated with the same demonstrative rigour, or it simply denotes a certainty obtained using a different method. Thus, the logic of the probable is identified with that branch of logic studying lines of reasoning through which probable conclusions can be produced and their degree of certainty measured.

The arguments brought into play in moral proofs are ordered according to two distinctions. The first of these differentiates between arguments that prove (*probantia*) and those that indicate (*indicantia*). The second distinguishes between arguments for the understanding, or principles (*Gründe*) and arguments for the will, or motives (*Beweggründe*). The former determine the degree of certainty of a judgement according to the probability that is attributed to it, while the purpose of the latter, whose field of application is human decisions (paragraph 230), is to guide an individual in his choices or to convince another person of the decision that he should make.[5] (This explains why the favoured domain of arguments for the will is the probability of testimony, the credibility of which must be evaluated in order to enlighten judicial decisions).

This is not a matter of separating two distinct types of judgement, but two uses of the same judgements, according to whether they are considered from the point of view of their cognitive function or their practical use in guiding a decision. Thus, Lambert's analysis falls within the decisionist tradition which, inheriting from Pascal, marked the probabilistic thinking of the 18th century, endeavouring to determine the conduct that it is reasonable to pursue in a situation of uncertainty, faced with the risks and potential advantages it presents (see Daston [8]). This is clearly expressed in Jacques Bernoulli's def-

[3] The manuscript of this course is kept in the Public and University Library of Geneva [6]. The probabilistic section is reproduced in volume 2/1 of the *Electronic Journal for History of Probability and Statistics* [7], accompanied by the article "Probabilité" from the *Encyclopédie* and the cross-reference table [17]. Did Lambert have knowledge of Cramer's text? There is no reason to believe so, as these were private lessons that remained unpublished until very recently. And although part of it was repeated in the article "Probabilité" in Diderot and d'Alembert's *Encyclopédie*, it was precisely not in its probabilistic presentation.

[4] For a more detailed analysis of this text, see [18].

[5] "Die Argumente *für* den Willen werden eigentlich da gebraucht, wo man untersucht, ob man sich zu etwas entschließen solle, oder auch, wo man andere dazu bereden will", ("Arguments for the will are used precisely when one seeks to know whether one should resolve to do something and also when one wishes to convince others to do it"), paragraph 230.

inition of the art of conjecture: “We define the *art of conjecture*, or *stochastic art*, as the art of evaluating as exactly as possible the probabilities of things, so that in our judgments and actions we can always base ourselves on what has been found to be the best, the most appropriate, the most certain, the best advised.”[6]

Lambert’s exposition of the theory of probable syllogisms is preceded by a classification of the different sorts of probabilities, on the basis of a series of interwoven distinctions. In particular, following Bernoulli, he distinguishes between *a priori* and *a posteriori* probabilities.

Probability can be calculated *a priori*, by the ratio of favourable cases over possible cases, from the moment that all cases are equally possible and that the number of possible cases results from the very nature of the probabilised event. This first category covers the field of games of chance and random draws where the number of favourable cases results from the rules defining the conditions of the game or draw. However, this *a priori*-calculated probability only concerns theoretical game models, not games that are really played, because the latter, Lambert specifies, require us to take into account irregularities affecting the physical properties of the instruments used (paragraph 152). It should be noted that the analysis of this first category is limited to three paragraphs (three pages) and has two main purposes. The first is to show that probability can be taken in two senses, inasmuch as the probable can be contrasted on the one hand to contrast with the *necessary*, “in relation to the thing itself”, and on the other hand to contrast with the *certain*, in that it is not possible for us to predict which event will happen at each test. Without using these terms, and without dwelling on the question particularly, Lambert thus introduces a distinction between an objective and a subjective form of probability. The second purpose of this analysis is to discuss the applicability of the principle of the equipossibility of cases.

The second category of probability distinguished by Lambert is *a posteriori* probability (paragraphs 153–161). However, this does not involve measuring the probability of an event’s occurrence on the basis of its observed past frequency, but determining the probability of an empirical proposition on the basis of the frequency of its confirmation. For Lambert’s analysis is situated within the field of logic, and the subject of his analysis is therefore epistemic probability. It is worth noting that Lambert makes a direct transition from the observed frequency of confirmation of the proposition to the measurement of its degree of probability; he passes over the question of the nature of the relation between frequency and probability. He simply indicates that the ratio of the number of cases confirming the proposition over the total number of cases “determines, in accordance with nature, the degree of probability of the proposition” (“*Diese Verhältnis bestimmt den Grad der Wahrscheinlichkeit des Satzes der Natur gemäß*”, paragraph 154). This is because his aim is not to give an empirical meaning to the concept of probability, nor to study its relation to the field of phenomena, but to provide the instruments allowing to combine the degrees of certainty attributed to propositions in order to determine the degree of certainty of the conclusion.

However, Lambert takes care to indicate how the observed frequencies should be measured, specifying notably that the exactness of the measurement of frequency increases with the number of observations, and that it is all the more necessary to use a large number of observations when the propositions are more particular. (So, for example, he explains that we should not content ourselves with calculating the number of deaths in a year; we should take into account age, type of illness, month of the year, etc. to construct particular classes of observations, and since the number of cases in each class will then be smaller, it is important to carry out a large enough number of observations, paragraph 156).

Finally, the third category distinguished by Lambert is that of the probability of inductions, by means of which we evaluate the probability of causes based on observation of the consequences. These consequences are of two sorts (paragraph 165). Firstly, there are physical consequences, i.e. events, for which we can measure the probability of the conditions that make them possible. Secondly, there are logical consequences, which do not deal with the events themselves, but with the concepts and propositions formed about them. For this reason, logical consequences are more general, and in lines of reasoning they include the former as a particular class. The theory of probable syllogisms can then be used to explain how to establish the probability of a hypothesis on the basis of its consequences.

4. The theory of probable syllogisms

The theory of probable syllogisms applies to particular propositions, and it allows to go beyond their indetermination to determine the degree of certainty of the conclusion. So, for example, the proposition “some A are B” implies that there are some cases where the predicate B belongs to A and other cases where it does not. Consequently, if we can enumerate the possible cases, differentiating between those that verify the proposition and those that do not, then we can determine the ratio of the number of favourable cases over the number of possible cases, or, as Lambert puts it, the ratio of “cases that fit the proposition” over the total number of cases.

More precisely, Lambert puts it like this: thanks to the enumeration of favourable and unfavourable cases, we can determine “the ratio between the cases that fit and those that do not fit the proposition”, enabling us to move from a particular indeterminate proposition to a determinate proposition, indicating how many A are B: “we know not only that *some* A are B, but more exactly how many are and how many are not”, he writes.[7] If Lambert puts it like this, we may imagine that it is

[6] “*Ars Conjectandi* sive *Stochastice* nobis definitur ars metiendi quam fieri protect exactissime probabilitates rerum, eo fine, ut in judiciis and actionibus nostris semper eligere vel sequi possimus id, quod melius, satius, tutius aut consultius fuerit deprehensum” [3, Chapter II].

[7] “Man weiß nicht nur, daß *etliche* A, B sind, sondern genauer, wie viele es sind, und wie viele es nicht sind”, paragraph 154.

because, within a syllogistic framework dominated by the predicative form, favourable cases are identified with the help of the relation of belonging, i.e. according to whether or not the subject possesses a given property. Now, as we have seen, this relation "determines the degree of probability of the proposition" (paragraph 154).

In fact, Lambert is making use of a double equivalence here:

Firstly, he identifies the ratio of the number of cases that verify the proposition over the number of cases that do not with the probability of the event. In other words, he does not identify the *frequency* of the event with the probability, but what we might call the *chance* of the event. Secondly, he then identifies this probability with the probability of the proposition, in other words with its degree of certainty.

Why does Lambert reason in terms of chance rather than frequency? Because the model underlying his analysis is that of the lottery, and therefore of gambling. This appears explicitly in the text: Lambert draws an analogy between the ratio of the number of cases that verify a proposition over the number that do not and the ratio of the number of winning tickets over the number of losing tickets in a lottery. In this way, he likens the arguments of the syllogism to the different sets of tickets in a lottery. He posits firstly that there are as many arguments as there are batches of tickets, and secondly that for each argument, the ratio of the number of cases that verify the proposition over the number of cases that do not is identical to the number of winning tickets over the number of losing tickets in the lottery. He deduces that the calculation of the degree of probability of an argument is comparable to the calculation of the degree of probability of drawing a winning ticket. By combining particular determinate propositions and indeterminate propositions, it is therefore possible to calculate the degree of probability of the conclusion, just as one can calculate the probability of drawing a winning ticket.

This is what Lambert demonstrates in the analysis that follows. Let us consider the two true premises ($\frac{3}{4}$ A are B) and (C is A), where C denotes an individual (paragraph 189). These two premises express a relation of belonging: the major premise indicates that $\frac{3}{4}$ of A possess the predicate B, and the minor premise that C belongs to the set of A. The question is then to determine to what extent it is possible to conclude that C possesses the predicate B.

If we knew that C belonged to the $\frac{3}{4}$ of A that possess the predicate B or to the $\frac{1}{4}$ of A that do not, we could answer the question with certainty. As we do not have that information, we are obliged to remain within the realm of the probable, and we must accept that it is three times more probable that C belongs to the set of A that possess the predicate B than to the set of A that do not.[8]

It follows that the proposition that C possesses the predicate B is not certain, but has a probability of $\frac{3}{4}$, which Lambert expresses in the form:

$$\text{C}\frac{3}{4}\ \text{is B.}$$

This expression is derived from the previous one by shifting the fraction $\frac{3}{4}$, which no longer applies to the subject, but to the copula. In other words, it is no longer a content – the number of A possessing B – that is measured, but the modality according to which the proposition expressing that content is affirmed. Consequently, it is not the possibility of occurrence of the event "A possesses the predicate B" that is measured, but the degree of certainty attributed to the proposition. The probability does indeed measure a degree of certainty, not a physical possibility.

The reasoning is then expressed in the form of the following syllogism, thanks to which the numerical coefficient is transferred from the major premise to the conclusion,[9] and is no longer applied to part of a set, but to a degree of certainty:

$$\frac{3}{4}\ \text{A are B}$$
$$\text{C is A}$$
$$\text{therefore C}\ \frac{3}{4}\ \text{is B.}$$

This conclusion should then be read "C is B to the degree $\frac{3}{4}$".

If, now, C is no longer an individual but a type or species, so that the minor premise is no longer particular but universal, the conclusion also becomes universal, and the degree of probability of the conclusion remains unchanged:

$$\frac{3}{4}\ \text{A are B}$$
$$\text{All C is A}$$
$$\text{All C}\ \frac{3}{4}\text{is B.}$$

The situation is parallel when the minor premise is indeterminate. We obtain:

[8] Lambert expresses this by writing that "it is three times more probable that C is among those A that are B than among those that are not": "es ist 3mal vermutlicher, daß C unter den A sei, die B sind, als aber unter denen, die es nicht sind" (paragraph 189).

[9] Lambert writes that in the syllogism: "the degree that determines the probability moves over from the major premise to the conclusion" ("... zieht sich der Grad, so die Wahrscheinlichkeit bestimmt, aus dem Obersatz in den Schlußsatz", paragraph 191).

$$\frac{3}{4}\ \text{A are B}$$
$$\text{some C are A}$$
$$\text{some C}\ \frac{3}{4}\ \text{are B.}$$

Finally, when the minor premise is a particular determinate proposition, we obtain a probable particular determinate conclusion. For example,

$$\frac{3}{4}\ \text{A are B}$$
$$\frac{2}{3}\ \text{C are A}$$
$$\frac{2}{3}\ \text{C}\ \frac{3}{4}\ \text{are B.}$$

Thus, we can see that the quantity of the proposition is transferred from the minor premise to the conclusion whereas the degree of certainty is transferred from the major premise to the conclusion.

Lambert then shows how the degree of probability can be transferred from the minor premise to the conclusion (paragraph 191). Let MNPQ be the characters forming the intension of the concept B, where we do not know if any of these characters are specific to B. Let us assume that we have the following propositions:

$$\text{MNPQ is B}$$
$$\text{C is MNP.}$$

Since we do not know whether C also possesses the predicate Q, we can only deduce from the above premises that it is probable that C is B. The question is then to determine the degree of probability of this conclusion.

Let us assume that we know that:

$$\text{MNPQ} = \text{A}$$
$$\text{MNP} = \frac{2}{3}\ \text{A},$$

we can then form the following syllogism:

$$\text{All A is B}$$
$$\text{C is}\ \frac{2}{3}\ \text{A}$$
$$\text{Therefore C}\ \frac{2}{3}\ \text{are B.}$$

If we now combine (paragraph 192) the two simple syllogistic forms above, in which the probability affects either the major or the minor premise, we can construct the following syllogism:

$$\frac{3}{4}\ \text{A are B}$$
$$\text{C is}\ \frac{2}{3}\ \text{A}$$
$$\text{therefore C}\ \frac{1}{2}\text{is B.}$$

This is because the probability of the conclusion that C is B is the product of the probability that C is A and the probability that A is B. In other words, this conclusion has in its favour $\frac{2}{3}$ of the $\frac{3}{4}$ of A that are B.

We can go further, by taking into account the fact that the propositions are affirmative, negative or indeterminate (paragraphs 193–198). Lambert denotes affirmative propositions by a, negative ones by e and indeterminate ones by u. For example, the formula

$$\left(\frac{2}{3}a + \frac{1}{4}e + \frac{1}{12}u\right)\ \text{A are B}$$

means that for all A, there are $\frac{2}{3}$ for which we know they possess the character B, $\frac{1}{4}$ for which we know they do not possess B and $\frac{1}{12}$ for which we do not know whether B belongs to them or not. (Note that this major premise constitutes a complete enumeration of A, because $\frac{2}{3} + \frac{1}{4} + \frac{1}{12} = 1$).

Likewise, the formula

$$\text{C is}\ \left(\frac{3}{5}a + \frac{2}{5}u\right)\ \text{A}$$

means that we know that $\frac{3}{5}$ of A belong to C and that, for the remaining $\frac{2}{5}$, we do not know whether they belong to C or not.

By taking the two propositions above as premises, we can construct the following syllogism:

$$\left(\frac{2}{3}a + \frac{1}{4}e + \frac{1}{12}u\right) \text{ A are B}$$
$$\text{C is } \left(\frac{3}{5}a + \frac{2}{5}u\right) \text{ A}$$
$$\text{therefore C}\left(\frac{2}{5}a + \frac{3}{20}e + \frac{9}{20}u\right) \text{ is B}$$

The conclusion is obtained by multiplying the coefficients together, and dividing the result into three classes: affirmative, negative and indeterminate. We then obtain the following result:

$$\begin{array}{l} \frac{2}{5}aa + \frac{3}{20}ae + \frac{3}{60}\ au \\ \qquad\qquad + \frac{4}{15}\ au^{10} \\ \qquad\qquad + \frac{2}{20}\ eu \\ \qquad\qquad + \frac{2}{60}\ uu \\ \hline \frac{2}{5}a + \frac{3}{20}e \quad + \frac{9}{20}\ u \\ \hline \end{array}$$

By reducing the terms to the same common denominator, we obtain the conclusion that, out of 20 cases, there are 8 affirmative, 3 negative and 9 indeterminate. In other words, for each case, there are 8 reasons to affirm the conclusion, 3 reasons to deny it and 9 reasons to leave it indeterminate.

Lambert then generalises his method, by showing how probabilistic syllogistics can be applied when the two premises are probable (paragraph 199). The degree of probability of the conclusion is then equal to the product of the probabilities of the premises. In modern terms, this means applying the rule of compound probabilities to the premises. For example, from the probable premises

$$\text{A } \frac{2}{3} \text{is B}$$
$$\text{C } \frac{3}{4} \text{ is A,}$$

we obtain the conclusion

$$\text{C } \frac{1}{2} \text{ is B.}$$

Lastly, Lambert extends probabilistic syllogistics: (i) to chains of syllogisms, where the probability of the conclusion is obtained by multiplying the fractions attributed to both the copulas and the middle terms, and (ii) to other figures of syllogism.

5. Application to the probability of testimony

Before concluding his study with a comparative analysis of the different forms of certainty-geometrical, physical and moral (paragraphs 244–265)-, Lambert applies the syllogistic approach to the probability of testimony, which provides instruments for the quantitative measurement of historical certainty.

Lambert's application of probability theory to the field of testimony was not original. On the contrary, the probabilistic tradition had been active in this field since the end of the 17th century. Historians of probability agree that study of the probability of testimony started in Great Britain with the works of George Hooper [9] and John Craig [5], both published in 1699.

[10] The original text contains $\frac{4}{12}$, which is obviously an erratum.

Hooper and Craig applied their analyses to three different forms of testimony: isolated, simultaneous and successive. Simultaneous testimonies are particularly prominent in the judicial field, successive testimonies in the epistemology of historical knowledge, notably with regard to biblical narratives.

Studies of the probability of testimony have been put to the service of two major preoccupations: theological among the British authors (Hooper and Craig, but also Richard Price), legal and historical in the French tradition, marked notably by the works of Bicquilley, Condorcet and Laplace, and in the Germanic tradition of Jacques and Nicolas Bernoulli and Lambert. It was then associated with another field of probabilism with which it entertained close relations, the probability of judgements, which was the subject of important (but controversial) studies, especially by Condorcet, in the 18th century,[11] and then of further studies during the 19th century by Laplace [14, pp. 455–470] and [15], Lacroix [11, nos. 140–150], Poisson [20, nos. 36–40] and Cournot [4, paragraphs 224–225].

Craig and Hooper established, in particular, two laws of variation in the degree of validity of testimony as a function of the number of witnesses. The first is the law of increasing probability of testimony as a function of the number of direct (simultaneous) witnesses; the second is the law of decreasing probability of testimony as a function of the number of indirect (successive) witnesses.[12] The work of Craig and Hooper was known to continental probabilists, but Lambert's objective here did not concern the measurement of the probability of *one* testimony as a function of the number of witnesses: his aim was to define the rules for the *combination* of testimonies. His analysis focuses on the measurement of the probability of a testimony resulting from the combination of several testimonies, calculated on the basis of the credibility of independent witnesses and according to whether their testimonies concur or conflict. Lambert's objective is to develop a general formula for the calculation of the credibility of combined testimonies.

When applied to testimony, the measurement of the probability of judgements concerns the credibility of information (*die Glaubwürdigkeit der Nachrichten*). The question then centres on the mode of combination of particular credibilities to which different degrees of certainty are attributed. The difficulty held to be inherent in trying to judge the general credibility of a witness justifies Lambert's decision to identify the credibility of a witness with the credibility of a particular testimony. In addition, he posits that the degree of probability of historical information increases with the number of witnesses, provided that the witnesses are independent and not chosen. Each independent witness can then be considered as a particular argument independent from the others. The effect of combining arguments is either to strengthen the information when the arguments concur or, on the contrary, to weaken it if they conflict. Lambert analyses the case where the independent witnesses uphold similar judgements, and then the case where they uphold different judgements.

5.1. Analysis of concurring testimonies

Lambert posits that the credibility of a testimony is measured by the number of truths, lies and non-truths (i.e. indeterminate propositions) that the account contains. This three-way division then allows to transpose the categories of affirmative, negative and indeterminate propositions used previously to the study of the credibility of testimony.

Taking up the above notations again, the credibility of a witness (witness A) stating ten truths, three non-truths and one lie is expressed by the following formula:

$$10a + 3u + 1e.$$

This formula signifies that out of 14 judgements set forth in the testimony, 10 are clearly true, 1 is clearly false and 3 are indeterminate. To put it another way, out of the 14 judgements expressed, we should believe the witness in 10 cases, believe the opposite in one case and not believe either in three cases.

If we now consider a second witness (witness B), whose credibility is equal to

$$12a + 5u + 2e,$$

we can calculate the credibility corresponding to the sum of the two accounts, by multiplying these two credibilities together:

$$120aa + 86au + 15uu + 11eu + 2ee + 32ae.$$

This formula can then be simplified by carrying out the following operations:

1. We can eliminate judgements of the form *ae*, because they correspond to a contradictory belief, namely a simultaneous belief in the truth of the judgement of witness A and in the truth of the contrary judgement uttered by witness B.
2. We can add together judgements of the form *au* and judgements of the form *aa*, because although the former are indeterminate for witness B, they are recognised to be true in the testimony of witness A.
3. Lastly, we can add together judgements of the form *eu* and judgements of the form *ee* by applying the same principle (but negatively).

[11] For an analysis of the probability of testimony in the 18th century, particularly in Condorcet, Laplace and Hume, see the article by Zabelle [24].

[12] This second law is taken up in the Anglo-Saxon philosophical tradition by Locke [16, bk. IV, Chapter 16] and Hume [10, bk. I, part. III, Section 13].

We thus obtain a credibility for the two testimonies combined, expressed by the formula:

$$206a + 15u + 13e.$$

We can extend this method of calculation to the case where the testimony results from the combination of the accounts of three witnesses, simply by multiplying the above formula with that expressing the credibility of the third witness, and so on for any further number of witnesses. Lambert then proposes the general formula of the calculation:

$$\begin{aligned}&\text{Credibility of witness A}: \quad \mathrm{M}a + \mathrm{N}u + \mathrm{P}e\\&\text{Credibility of witness B}: \quad \mathrm{m}a + \mathrm{n}u + \mathrm{p}e\\&\text{Credibility of two witnesses}: (\mathrm{Mm} + \mathrm{Mn} + \mathrm{mN})a + \mathrm{Nn}u + (\mathrm{Pp} + \mathrm{Pn} + \mathrm{pN})e.\end{aligned}$$

5.2. Analysis of divergent testimonies

This situation can take one of two distinct forms:

(a) Either the two testimonies express entirely different judgements, in which case no calculation of compound credibility is possible, because there is no common measure between the judgements.
(b) Or the judgements are contradictory. In this case, we simply carry out the same calculation as above, but reversing the values of credibility of the conflicting testimony, in other words inverting the values of judgements of the forms *a* and *e*. For example, Lambert writes (paragraph 238), if the second witness expresses testimony contrary to that of the first witness, his credibility will no longer be of the form:

$$12a + 5u + 2e,$$

but

$$2a + 5u + 12e.$$

By combining this credibility with that of the first witness (whose credibility is $10a + 3u + 1e$), we obtain:

$$76a + 15u + 53e.$$

As Lambert observes (paragraph 239), this method of calculating credibility is not specific to the analysis of testimony. It can be applied equally well to the measurement of the strength with which an argument supports a proposition. Thus, an argument of the form

$$12a + 5u + 2e$$

is an argument that proves the proposition (therefore affirms it) in 12 cases, does not prove it (therefore leaves it undecided) in 5 cases and proves the contrary or refutes it in 2 cases. There is a three-way partition of arguments into those that prove the proposition, those that prove the contrary and those that do not prove anything, corresponding to the three-way division of propositions into affirmative, negative and indeterminate ones. Consequently, when the conclusion of a series of probable syllogisms is of the form

$$\text{All A}\left(\frac{12}{19}a + \frac{5}{19}e + \frac{2}{19}u\right)\text{ is B},$$

the coefficients applied to the copula measure the credibility of the proposition "All A is B", and therefore, the strength of the argument.

This development gives Lambert the opportunity to show that his analysis has the advantage of being more widely applicable than that by which Jacques Bernoulli, in *Ars conjectandi*, distinguished between *pure* arguments (which in part prove a proposition and in part do not prove it) and *mixed* arguments (which in part prove the proposition and in part prove the contrary).[13] For more detail about the respective approaches of Bernoulli and Lambert, see the article by Shafer [22], examining the positions of these two scholars in the treatment of non-additive probabilities and showing their relations to the Dempster–Shafer theory.

6. Conclusion

As Shafer notes [22, p. 362], Lambert's theory of combinations of probable arguments, using non-additive probabilities, could not be understood in the 18th century (and even less so, we might add, in the 19th century). So it is not surprising

[13] The example Bernoulli uses is that of a crime committed in a crowd, where we know that the author of the crime was wearing a black coat. Now Gracchus, together with three other individuals, was wearing a coat of this colour. The fact of wearing a black coat is an argument proving the guilt of Gracchus, but it is a mixed argument, because in 1 case it accuses Gracchus, but in 3 other cases it accuses the others. If Gracchus now turns pale during questioning, this paleness is a pure argument, because, explains Bernoulli, "it proves the guilt of Gracchus, if it is caused by a troubled conscience; but conversely it does not prove his innocence, if it has another cause", Bernoulli [3, p. 218,19, p. 30].

that it has not marked the history of probability calculus. But we can add that the syllogistic form turns out to be artificial when applied to probabilistic reasoning, constituting more of an obstacle than an instrument at the service of the probabilistic approach. On the other hand, it is clear, as we have seen, that Lambert only reserves a minor role for what we would now call frequentist probability. His aim is absolutely not to measure the possibility of a random event occurring by means of a calculation of frequencies; his concern is to characterise and formalise probable reasoning, and then to apply it to the study of the probability of testimony. His analysis is therefore deployed on the level of epistemic probability, confirming the preference of early probabilists for the epistemic interpretation of probability.

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